Global Aspects of the WZNW Reduction to Toda Theories*

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Abstract. It is well-known that the Toda Theories can be obtained by reduction from the Wess-Zumino-Novikov-Witten (WZNW) model, but it is less known that this WZNW $\to$ Toda reduction is 'incomplete'. The reason for this incompleteness being that the Gauss decomposition used to define the Toda fields from the WZNW field is valid locally but not globally over the WZNW group manifold, which implies that actually the reduced system is not just the Toda theory but has much richer structures. In this note we furnish a framework which allows us to study the reduced system globally, and thereby present some preliminary results on the global aspects. For simplicity, we analyze primarily $0 + 1$ dimensional toy models for $G = SL(n, \mathbf{R})$, but we also discuss the $1 + 1$ dimensional model for $G = SL(2, \mathbf{R})$ which corresponds to the WZNW $\to$ Liouville reduction.

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1. Introduction

In recent years the subject of integrable models in $1 + 1$ dimensions, especially conformally invariant ones, has been attracting considerable attention. Among them are the standard Toda theories governed by the Lagrangian

$$\mathcal{L}_{\text{Toda}}(\varphi) = \frac{\kappa}{2}\left[\sum_{i,j=1}^{l}\frac{1}{2|\alpha_i|^2}K_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j - \sum_{i=1}^{l}m_i^2\,\exp\left(\frac{1}{2}\sum_{j=1}^{l}K_{ij}\varphi^j\right)\right], \qquad (1.1)$$

where κ is a coupling constant, K_{ij} is the Cartan matrix and the α_i are the simple roots of the simple Lie algebra $\mathcal{G}$ of rank l. These Toda theories have been studied intensively over the past several years, with particular reference to an application to two dimensional gravity, since the Liouville theory emerges when the underlying group G, for which $\mathcal{G} = \text{Lie}(G)$, is $SL(2,\mathbf{R})$. One of the salient features of the Toda theories is that they possess as symmetry algebras so-called $\mathcal{W}$-algebras [1], which are a polynomial extension of the chiral Virasoro algebra. It has been by now well recognized [2] that both the origin of the $\mathcal{W}$-algebras and the integrability of the Toda theories can be nicely understood by reducing the Toda theories from the Wess-Zumino-Novikov-Witten (WZNW) model [3]*,

$$S_{\text{WZ}}(g) = \frac{\kappa}{2}\int d^2x\,\eta^{\mu\nu}\,\text{tr}\,(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) - \frac{\kappa}{3}\int_{B_3}\text{tr}\,(g^{-1}dg)^3\ . \qquad (1.2)$$

The reduction is performed in the Hamiltonian formalism by imposing a certain set of first class constraints in the WZNW model, where the connection between the WZNW field $g \in G$ in (1.2) and the Toda fields φ^i in (1.1) arises from the Gauss decomposition,

$$g = g_+ \cdot g_0 \cdot g_- \ . \qquad (1.3)$$

Here $g_{0,\pm}$ are from the subgroups $e^{\mathcal{G}_{0,\pm}}$ of G where $\mathcal{G}_0$ is the Cartan subalgebra and $\mathcal{G}_\pm$ are the subalgebras consisting of elements associated to positive or negative roots — hence the corresponding decomposition in the algebra being $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_-$. Then the Toda fields are given by the middle piece of the Gauss decomposition,

$$g_0 = \exp\left(\frac{1}{2}\sum_{i=1}^{l}\varphi^i H_i\right), \qquad (1.4)$$

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* The space-time conventions are: $\eta_{00} = -\eta_{11} = 1$, $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ and $\partial_\pm = \partial_0 \pm \partial_1$. The WZNW field g is periodic in x^1 with period 2π.

where $H_i \in \mathcal{G}_0$ are the Cartan generators associated to the simple roots α_i.

However, the above picture of the WZNW $\rightarrow$ Toda reduction is not quite complete because the Gauss decomposition (1.3) is valid only locally in the neighbourhood of the identity $g = 1$ but not globally over the group manifold G. (This incompleteness has been noticed already in the early work on the WZNW $\rightarrow$ Liouville reduction [4].) This suggests that, in general, the reduced theory may be thought of as a system consisting of several subsystems defined on each of the patches introduced to cover the entire group manifold, and that the Toda theory is merely the subsystem given on the Gauss decomposable patch where (1.3) is valid.

Clearly, for understanding what the WZNW reduction really brings about we need to know (i) what is a possible general framework for a global description of the reduced system, and (ii) what are the physical implications of 'being global', rather than just 'being local' considering the Toda theory only. The purpose of this paper is to set out an investigation toward these desiderata. In order to elucidate the essence as well as to ease the problem, we simplify the situation by considering primarily the toy models given by the same reduction in $0 + 1$ dimension, that is, we just neglect the spatial dimension in the usual WZNW $\rightarrow$ Toda reduction. By doing so, the WZNW model becomes a system of a particle moving freely on the group manifold G, and the reduction renders the reduced configuration space essentially flat with a diminished dimension, giving rise to a Toda type potential. For brevity, we call those reduced toy models '0 + 1 dimensional Toda theories', and set $\kappa = 1$ throughout.

The plan of the present paper is the following: We first illustrate in sect.2 an interesting physical effect caused by being global in a simple setup, where we take up a $0 + 1$ dimensional Toda theory for $G = SL(2, \mathbf{R})$, *i.e.*, the Liouville toy model. We shall observe that a locally catastrophic motion of a particle can be interpreted as an aspect of a globally stable motion, an oscillation. Then in sect.3 we provide a framework for $G = SL(n, \mathbf{R})$ which enables us to discuss the reduction globally. This will be done in two ways — first by furnishing a decomposition (Bruhat decomposition) to cover the entire group manifold, and second by giving a gauge fixing (vector Drinfeld-Sokolov gauge) which is convenient for arguing the (dis)connectedness of the reduced phase space. In sect.4 we move on to the actual $1 + 1$ dimensional case for the WZNW $\rightarrow$ Liouville reduction, where we shall see even more interesting physical implications that are missing in the $0 + 1$ dimensional toy model. We find, for instance, that in the global

point of view, the singularities in the classical Liouville solution are nothing but the points where the Gauss decomposition breaks down, and the number of those points can be regarded as a conserved topological charge. All of our analyses in this paper are purely classical. Sect.5 will be devoted to our conclusions and outlooks.

2. $SL(2, \mathbf{R})$ Toda (Liouville) theory in $0 + 1$ dimension

For a simple illustration, we shall begin with a toy model where the aspects of being global appear dramatically. We first define our $0 + 1$ dimensional model in the Hamiltonian formalism, and then provide a heuristic argument signifying the global aspects in the Lagrangian formalism. Later we return to the Hamiltonian formalism to examine the model in more detail.

2.1. Hamiltonian reduction

Consider a point particle moving on the group manifold G. The description of the model as a Hamiltonian system is standard: the phase space is the cotangent bundle of the simple Lie group G,

$$M = \mathrm{T}^*G \simeq \{(g, J) \mid g \in G, \ J \in \mathcal{G}\}, \tag{2.1}$$

where the fundamental Poisson brackets are $\{g_{ij}, g_{kl}\} = 0$ and

$$\{g_{ij}, \mathrm{tr}(T^a J)\} = (T^a g)_{ij}, \quad \{\mathrm{tr}(T^a J), \mathrm{tr}(T^b J)\} = \mathrm{tr}([T^a, T^b] J), \tag{2.2}$$

with T^a being a basis set of matrices in some irreducible representation of $\mathcal{G}$. The Hamiltonian is then

$$H = \frac{1}{2} \mathrm{tr} J^2, \tag{2.3}$$

which yields the dynamics,

$$\dot{g} = \{g, H\} = Jg, \qquad \dot{J} = \{J, H\} = 0. \tag{2.4}$$

Hence the particle (whose position is given by the value $g(t)$ on G) follows the free motion on the group manifold, $\frac{d}{dt}(\dot{g}g^{-1}) = 0$. Incidentally, we note that the 'right-current',

$$\tilde{J} := -g^{-1} J g, \tag{2.5}$$

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commutes with the 'left-current' J and forms the Poisson brackets analogous to (2.2),

$$\{g_{ij}, \text{tr}(T^a \tilde{J})\} = -(gT^a)_{ij}, \quad \{\text{tr}(T^a \tilde{J}), \text{tr}(T^b \tilde{J})\} = \text{tr}([T^a, T^b]\tilde{J}), \qquad (2.6)$$

Now let us take $G = SL(2, \mathbf{R})$ with its defining representation for (2.2). The constraints we will impose are precisely the same as those imposed in the WZNW $\rightarrow$ Liouville reduction [4],

$$\text{tr}(e_{12}J) = \mu \qquad \text{and} \qquad \text{tr}(e_{21}\tilde{J}) = -\nu, \qquad (2.7)$$

where μ and ν are constants, and e_{ij} are the usual matrices having 1 for the (i, j)-entry and 0 elsewhere. It is almost trivial to see that the two constraints in (2.7) are first class.

To be more explicit, let us parametrize the phase space M as

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} j_0 & j_+ \\ j_- & -j_0 \end{pmatrix}. \qquad (2.8)$$

Then the first constraint in (2.7) is just $j_- = \mu$ while the second implies

$$\mu g_{12}^2 - 2j_0\, g_{22}g_{12} - j_+ g_{22}^2 = -\nu. \qquad (2.9)$$

From this we see that on the hypersurface $g_{22} = 0$ in the constrained submanifold $M_{\text{c}} \subset M$ defined by (2.7) we must have $g_{12} = \sqrt{-\nu/\mu}$ if $\mu\nu < 0$. But if $\mu\nu > 0$, then there is no solution for g_{12} on the hypersurface $g_{22} = 0$ in M_{c}, which implies that M_c, and consequently also the reduced phase space M_{red}, is *disconnected* — one with $g_{22} > 0$ and the other with $g_{22} < 0$. Recall that the Gauss decomposable patch is of the form (1.3), which for $SL(2, \mathbf{R})$ reads

$$g = g_+ \cdot g_0 \cdot g_- = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{x}{2}} & 0 \\ 0 & e^{-\frac{x}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}. \qquad (2.10)$$

It is then easy to see that on the Gauss decomposable patch we always have $g_{22} > 0$. Hence, when $\mu\nu > 0$ the conventional description of the reduced theory is actually sufficient (self-contained) as a description of the subsystem on the Gauss decomposable patch, as it is anyway decoupled from the subsystem for which $g_{22} < 0$. We shall discuss this case $\mu\nu > 0$ more generally in sect.3. In the rest of this section, however, we shall concentrate on the case $\mu\nu < 0$, where the importance of the global consideration is more transparent. (Below we set $\mu = -\nu = 1$ for simplicity.)

2.2. Lagrangian description — what is the fate of the particle?

We now illustrate — through a heuristic argument — an interesting global aspect of the reduced system. Consider first the Gauss decomposable patch (2.10). Observe that the local gauge transformations generated by the constraints (2.7) are

$$g \longrightarrow \alpha\, g\, \gamma, \qquad J \longrightarrow \alpha\, J\, \alpha^{-1}, \tag{2.11}$$

where $\alpha = e^{\theta e_{12}}$ and $\gamma = e^{\xi e_{21}}$. Thus in the Lagrangian formalism, the present constrained system is realized as a gauge theory [4,5] possessing the local symmetry under (2.11), with the help of Lagrange multipliers. Choosing the 'physical gauge' $a = c = 0$, one can eliminate the Lagrange multipliers using their equations of motion. This yields the effective, reduced Lagrangian,

$$L_{\mathrm{red}} = \frac{1}{4}\dot{x}^2 + e^x \ . \tag{2.12}$$

Thus, as long as we are on the Gauss decomposable patch, we obtain as the reduced subsystem a particle moving on a line under the influence of the exponential potential $V_{\mathrm{red}} = -e^x$. The equation of motion derived from L_{red} is hence

$$\ddot{x} = 2e^x, \tag{2.13}$$

which has, for 'energy' $E := \frac{1}{4}\dot{x}^2 - e^x < 0$, the general solution

$$x(t) = -2\ln\left(\frac{\cos\omega(t - t_0)}{\omega}\right), \tag{2.14}$$

where $\omega = \sqrt{|E|}$ and t_0 are constants determined from the initial condition given. For instance, for the initial condition $x(0) = 0$ and $\dot{x}(0) = 0$, we have $\omega = 1$, $t_0 = 0$. We then observe that the particle reaches the infinity $x = \infty$ with the finite time $\frac{\pi}{2}$. Hence, if this patch with $-\infty < x < \infty$ was the only 'world' for the particle, then the particle would sooner or later face a 'catastrophe', as long as the energy is negative. But fortunately, we know that there are other 'worlds' (patches) in the group manifold where the particle could live after it leaves the original patch. But then, what happens to the particle after it experiences the catastrophe?

To know this we first recall [4] that the entire $SL(2, \mathbf{R})$ group manifold can be covered by the four patches:

$$g = g_+ \cdot g_0 \cdot g_- \cdot \tau \qquad \text{with} \qquad \tau = \pm 1, \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.15}$$

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It is then straightforward to see (by proceeding similarly as before) that, on the two patches $\tau = \pm 1$, g_{22} never vanishes ($g_{22} > 0$ for $\tau = 1$ and $g_{22} < 0$ for $\tau = -1$) and moreover the reduced Lagrangian $L_{\rm red}$ takes the same form (2.11). Thus the hypersurface $g_{22} = 0$ is exactly the place where the local description using the two patches breaks down. (The other two patches, where g_{22} can become zero, admit neither a convenient gauge like the physical gauge nor a regular description at $g_{22} = 0$, as we will discuss shortly.) But since $g_{22} = 0$ is merely a lower dimensional submanifold in the entire group manifold, we shall for the moment disregard the singular hypersurface (the 'domain-wall' between the two good 'domains' $\tau = \pm 1$) and consider the motion of the particle only on the patches $\tau = \pm 1$.

We notice at this point that since the values of x in the two patches are defined seperately on each patch and also depend on the gauge fixing condition, we must provide a method to extract the 'physical position' of the particle which has gauge- and patch-independent meaning. This may be accomplished if we identify those g which are gauge equivalent under (2.11); namely, we define the physical position of the particle by the values of $g \in SL(2, R)/GL(1)_{\rm left} \times GL(1)_{\rm right}$. For instance, for $\lambda \neq 0$, the two points,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & \lambda^{-1} \end{pmatrix} \tag{2.16}$$

are gauge equivalent and hence may be regarded identical. Since in the patch $\tau = 1$ the physical gauge corresponds to the first one in (2.16) with $\lambda = e^{\frac{x}{2}}$, in the second gauge in (2.16) the trajectory of the particle moving from $x = 0$ to ∞ may be represented symbolically as

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{\infty} \end{pmatrix}. \tag{2.17}$$

In the patch $\tau = -1$, on the other hand, the physical gauge corresponds to $\lambda = -e^{\frac{x}{2}}$ and one can show by an analogous but 'converse' argument that when a particle appears at the catastrophic point $x = \infty$ it would take the converse path to the above and reaches the point $x = 0$ in the patch if it has a sufficient energy. In the new gauge this passage reads

$$\begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{\infty} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{2.18}$$

But the fact that the final point in (2.17) and the initial point in (2.18) are identcal (although that point does not belong to the two patches) implies that the particle *oscillates* with the period 2π between the two points $x = 0$ in the two patches.

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Clearly, what we need is a description of the redecued theory valid globally even on a patch which contains the catastrophic point. In fact, one can derive the reduced Lagrangian for any of the other two patches by choosing a gauge fixing condition properly. By doing this one finds that the reduced Lagrangian always takes the form (2.12) but it contains a singularity at the hypersurface $g_{22} = 0$. (Since g_{22} is gauge invariant, this statement is gauge independent.) This singularity stems from the fact that in the original configuration space G the gauge group $GL(1)_{\text{left}} \times GL(1)_{\text{right}}$ does not act freely on the hypersurface $g_{22} = 0$ and, as a result, the Lagrangian description of the reduced theory necessarily suffers from the singularity. By contrast, the gauge group does indeed act freely in the phase space $M = \text{T}^*G$ even on the hypersurface $g_{22} = 0$, it is therefore possible to have a globally well-defined description of the reduced theory using the Hamiltonian formalism. Next, we wish to find it out explicitly.

2.3. Oscillation in the Hamiltonian description

For a description of the reduced system in the Hamiltonian formalism, there can be two options; one by the Dirac approach (by introducing a gauge fixing condition and the Dirac brackets), and the other by the 'gauge invariant approach'. The latter begins with choosing a set of gauge invariant functions on M_{c} for a set of coordinates of the reduced phase space M_{red}, adopting the original Poisson brackets (2.1) as a basis for computing the reduced Poisson brackets. Since at the moment we do not have a convenient, global gauge fixing free from singularity we will pursue the latter approach.

Finding a gauge invariant basis set of functions on M_{c} would be easier if we could use a 'minimum' coordinate system of M_{red} which is, of course, of $\dim M_{\text{red}} = 2$. In the present case this could be achieved by eliminating four variables by solving (2.9) after setting $j_- = 1$ and using two gauge fixing conditions. This however leads to either non-polynomial expressions for gauge invariant functions or a singularity in g_{22} again. We shall circumvent this by not solving (2.9) explicitly but allowing an extra variable for the coordinate of M_{red} keeping (2.9) in mind. In this spirit we shall find a set of three gauge invariant functions on M_{c} having a relation among them so that $\dim M_{\text{red}} = 2$.

One can find easily such a set of functions which are invariant under the gauge transformations (2.11):

$$Q = g_{22}, \qquad P = g_{12} - j_0 g_{22}, \qquad H = j_0^2 + j_+. \qquad (2.19)$$

Here H is the Hamiltonian (2.3) on M_{c} whereas $P = \dot{Q} = \{Q, H\}$. The set (Q, H) is

actually enough to serve as a basis for gauge invariant functions $f(g, J)$ on M_c, because there exists a gauge fixing procedure which yields the two invariants (*i.e.*, the gauge in which $j_0 \to 0$, $\tilde{j}_0 \to 0$ with $\tilde{j}_0$ defined from $\tilde{J}$ analogously to j_0 in (2.8)). However, the above set does not form a 'polynomial basis' (*i.e.* in terms of which any polynomial gauge invariant function can be expressed polynomially), for P cannot be expressed polynomially in terms of them. In fact, the relation between the three invariants is

$$HQ^2 = P^2 - 1. \tag{2.20}$$

Thus we shall regard the two dimensional surface determined by (2.20) in the three dimensional space (Q, P, H) as the reduced phase space M_{red}. The variables of the space form a polynomial closed algebra under the Poisson brackets:

$$\{Q\,,P\} = \frac{Q^2}{2}, \qquad \{Q\,,H\} = P, \qquad \{P\,,H\} = QH. \tag{2.21}$$

Combining the last two equations in (2.21) and the fact that the Hamiltonian H is a constant of motion we find, for $H = E = -\omega^2 < 0$, that Q obeys the equation for a harmonic oscillator,

$$\ddot{Q} + \omega^2 Q = 0. \tag{2.22}$$

This result is consistent with (2.13) and (2.14) on account of (2.19) and (2.20). The oscillatory motion of the particle can also be seen in the phase space M_{red} if we slice the surface by a (negative) constant H, as it forms the ellipse, $P^2 + \omega^2 Q^2 = 1$.

We now provide a set of Hamiltonian subsystems to comprise the reduced Hamiltonian system on M_{red}. In terms of manifolds M_k, Poisson brackets $\{\,,\}_k$ and Hamiltonians H_k for $k = 1, 2, 3, 4$, the first and the third subsystems are given as

$$M_1 = \{\,(Q, P)\,|\,Q > 0, \quad -\infty < P < \infty\,\},$$
$$\{Q\,,P\}_1 = \frac{Q^2}{2}, \qquad H_1(Q, P) = \frac{1}{Q^2}(P^2 - 1), \tag{2.23}$$

and

$$M_3 = \{\,(Q, P)\,|\,Q < 0, \quad -\infty < P < \infty\,\},$$
$$\{Q\,,P\}_3 = \frac{Q^2}{2}, \qquad H_3(Q, P) = \frac{1}{Q^2}(P^2 - 1), \tag{2.24}$$

while the second and the fourth are

$$M_2 = \{\,(Q, H)\,|\,-\infty < Q < \infty, \quad HQ^2 + 1 > 0\,\},$$
$$\{Q\,,H\}_2 = \sqrt{HQ^2 + 1}, \qquad H_2(Q, H) = H, \tag{2.25}$$

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and

$$M_4 = \{\,(Q,H)\,|\, -\infty < Q < \infty, \quad HQ^2 + 1 > 0\,\},$$
$$\{Q\,,H\}_4 = -\sqrt{HQ^2 + 1}, \qquad H_4(Q,H) = H. \tag{2.26}$$

When these subsystems are glued together they constitute the reduced Hamiltonian system. The transition between the subsystems is done through the relation (2.20). In effect, the subsystem M_1 (resp. M_3) represents the $Q > 0$ (resp. $Q < 0$) part of the surface (2.20), and the subsystem M_2 (resp. M_4) represents the $P > 0$ (resp. $P < 0$) part, respectively. Thus we conclude that the reduced Hamiltonian system is perfectly well-defined in terms of the four local Hamiltonian subsystems.

3. $SL(n,\mathbf{R})$ Toda theory in $0+1$ dimension

In sect.2 we have seen by the $0 + 1$ dimensional toy model that the global viewpoint can change the interpretation of the dynamics drastically. In this section we give a general framework for $G = SL(n,\mathbf{R})$ to deal with the global reduced system, by generalizing the idea used in the previous section.

3.1. Global reduction — the Bruhat decomposition

Analogous to the previous $SL(2,\mathbf{R})$ case, the WZNW reduction to Toda theories is defined to the Hamiltonian system (2.1) – (2.3) by imposing a set of first class constraints. For $G = SL(n,\mathbf{R})$ these constraints are defined [2] by generalizing (2.7) as

$$\pi_-(J) = I_- \qquad \text{and} \qquad \pi_+(\tilde{J}) = -I_+, \tag{3.1}$$

where
$$I_- = \sum_{\alpha \in \Delta} \mu_\alpha E_{-\alpha}, \quad I_+ = \sum_{\alpha \in \Delta} \nu_\alpha E_\alpha \qquad \text{with} \quad \mu_\alpha \neq 0, \quad \nu_\alpha \neq 0 \,. \tag{3.2}$$

Here Δ is the set of simple roots, the μ_α, ν_α are *nonzero* constants associated to the step generators $E_{\mp\alpha}$, and the projections $\pi_\pm$ in (3.1) refer to the subalgebras $\mathcal{G}_\pm$ mentioned in sect.1. As before, these constraints generate a gauge symmetry of the type (2.11) with $\alpha \in e^{\mathcal{G}_+}$ and $\gamma \in e^{\mathcal{G}_-}$.

Since the obstacle for a global description is the intrinsic locality of the Gauss decomposition (1.3), it is natural to seek for a set of patches that cover the entire group

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manifold G having the Gauss decomposable patch in it. A natural choice is given by the *Bruhat* (or *Gelfand-Naimark*) *decomposition* [6],

$$g_m = g_+ \cdot m \cdot g_0 \cdot g_- \, , \tag{3.3}$$

where m is a diagonal matrix given, for $G = SL(n, \mathbf{R})$, by

$$m = \text{diag}\,(m_1, m_2, \ldots, m_n)\,, \qquad \text{with} \quad m_i = \pm 1, \quad \prod_i m_i = 1. \tag{3.4}$$

Obviously, there are 2^{n-1} possibilities for m. With these m the entire group manifold G can be decomposed as

$$G = \bigcup_m G_m \bigcup G_{\text{low}}, \tag{3.5}$$

where G_m corresponds to the 'domain' labelled by m whereas G_{low} is a union of 'domain-walls', *i.e.,* certain lower dimensional submanifolds of G. We note that these domains are disjoint, and the decomposition (3.3) of every $g \in G_m$ is unique. It is also worth noting that in this $G = Sl(n, \mathbf{R})$ case the union of the domains $\bigcup_m G_m$ is the open sub-manifold consisting of matrices with nonzero prinicipal minors whose signs are specified by m, which is possible in 2^{n-1} different ways for an $n \times n$ matrix of determinant 1 and nonzero minors. Correspondingly, G_{low} consists of matrices with unit determinant and at least 1 vanishing principal minor.

Because of the factor m in the decomposition (3.3), we obtain in general the reduced dynamics slightly different from the familiar Toda dynamics. More precisely, one can derive (by the conventional reduction procedure where one chooses the physical gauge) the reduced Lagrangian governing the dynamics on the patch G_m,

$$L_m = \frac{1}{2}\,\text{tr}\,\left(g_0^{-1}\dot{g}_0\right)^2 - \text{tr}\,\left(I_- g_0\, m\, I_+\, m^{-1} g_0^{-1}\right)\,. \tag{3.6}$$

For $m = \pm 1$, one sees upon using (1.5) and $K_{ij} = \frac{|\alpha_i|^2}{2}\text{tr}(H_i H_j)$ that the Lagrangian (3.6) reduces to the standard Toda Lagrangian (1.1) in $0 + 1$ dimension; otherwise it differs from the standard one in general. Thus the reduced system consists of many 'quasi-Toda' subsystems, among which the standard Toda appears on the trivial patch $m = 1$. Symbolically, we may therefore write the actual reduction as

$$\text{WZNW} \longrightarrow \text{Toda} \oplus (\text{Toda})' \oplus (\text{Toda})'' \oplus \cdots \, . \tag{3.7}$$

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We are now interested in the question whether these different subsystems are discon-
nected or not.

3.2. Vector Drinfeld-Sokolov gauge

In sect.2 we have seen, based on the consistency of the constraint (2.8) against the
condition $g_{22} = 0$, that the reduced phase space $M_{\rm red}$ is disconnected if $\mu\nu > 0$. Now
we shall try to examine the (dis)connectedness generally for $SL(n, \mathbf{R})$. A convenient
method for doing this is to employ, instead of the physical gauge, the so-called Drinfeld-
Sokolov (DS) gauge [7] *both* for the left and right currents. (The DS gauge has been used
to obtain a $\mathcal{W}$-algebra basis in the WZNW (Kac-Moody) reduction [2].) For simplicity,
in the rest of this section we just consider the case where all the μ's and ν's are positive
and set all of them to unity.

We call the gauge *vector DS gauge* for $SL(n, \mathbf{R})$ if the left and right currents are
in the form,

$$J = I_- + \sum_{i=2}^{n} u_i e_{1i} \qquad \text{and} \qquad \tilde{J} = -I_+ - \sum_{i=2}^{n} u_i e_{i1}. \tag{3.8}$$

It is clear that since the usual, 'chiral' DS gauge is well-defined (*i.e.*, it is attainable and
specifies a unique representative among the gauge equivalent points in the phase space)
for a chiral Kac-Moody current under the same setting of gauge group, the above vector
DS gauge is also well-defined. Now, it follows from (2.5) and the second equation in
(3.8) that the matrix g has the same value along each anti-diagonal line:

$$g_{ij} = g_{kl}, \qquad \text{if} \quad i + j = k + l. \tag{3.9}$$

It also follows that

$$g_{1,j-1} = \sum_{i=2}^{n} u_i g_{ij}, \qquad \text{for} \quad j = 2, \ldots, n, \tag{3.10}$$

which shows that all the 'lower' entries $g_{1,j-1}$ in the first row for $1 \le j - 1 \le n - 1$ can
be expressed in terms of the 'higher' ones together with u_i's for $2 \le i \le n$. Thus, if we
set

$$u_{i+j} := g_{ij}, \qquad \text{for} \quad n + 1 \le i + j \le 2n, \tag{3.11}$$

then, combining with the first equation of (3.8), the reduced phase space $M_{\rm red}$ is
parametrized by the set of variables, $u_2, u_3, \ldots, u_{2n}$, which are subject to the condition

$\det g = 1$. Hence the total dimension of the reduced phase space is $\dim M_{\mathrm{red}} = (2n - 1) - 1 = 2(n - 1)$. Accordingly, in the vector DS gauge the matrix g can be written as

$$
g = \begin{pmatrix}
g_{11}(u) & g_{12}(u) & \cdots & g_{1,n-1}(u) & u_{n+1} \\
g_{12}(u) & g_{13}(u) & \cdots & u_{n+1} & u_{n+2} \\
\vdots & \vdots & & \vdots & \vdots \\
g_{1,n-1}(u) & u_{n+1} & \cdots & u_{2n-2} & u_{2n-1} \\
u_{n+1} & u_{n+2} & \cdots & u_{2n-1} & u_{2n}
\end{pmatrix} . \tag{3.12}
$$

However, the objects of our concern are not the u_i's now, but their gauge invariant functions defined by the principal minors of the matrix g,

$$
Q_n := u_{2n}, \qquad Q_{n-1} := \det \begin{pmatrix} u_{2n-2} & u_{2n-1} \\ u_{2n-1} & u_{2n} \end{pmatrix}, \qquad Q_{n-2} := \cdots, \tag{3.13}
$$

for Q_i with $i = 2,\ 3,\ \ldots,\ n$, that is, those $n - 1$ principal minors constructed from the lower right corner of the matrix g. It is not difficult to see that they are gauge invariant under (2.11). On the Gauss decomposable patch the Toda variables φ^i in (1.5) are directly related to those principal minors by $e^{-\varphi^i/2} = Q_{i+1}$, but unlike the Toda variables these Q_i are globally well-defined over the entire reduced phase space. These Q_i are a generalization of the Q variable used in the previous section.

3.3. Is Heaven connected with Hell?

As in the case of $SL(2, \mathbf{R})$, we want to know the global structure of the reduced phase space in the $SL(n, \mathbf{R})$ case. Although the entire global structure for a generic n seems hard to know, we can at least ease the problem by restricting ourselves to the simpler question whether the standard Toda theory, that is, the reduced subsystem on the Gauss decomposable patch, is disconnected from the rest of the subsystems or not.

More precisely, we shall ask whether there exists a smooth path connecting the domain,

$$
Q_2 > 0, \quad Q_3 > 0, \quad \cdots, \quad Q_n > 0, \tag{3.14}
$$

and the remaining domains of the reduced phase space. (We note that for a manifold 'connectedness' and 'path-connectedness' are the same.) We call the domain (3.14) — which is the domain for the standard Toda subsytem — simply 'Heaven', and the remainder 'Hell'. Suppose now that there exists such a path connecting Heaven and Hell. Let t be a real parameter of the path, and t_0 the time passing the border between Heaven and Hell,

$$
Q_2(t_0)\, Q_3(t_0) \cdots Q_n(t_0) = 0. \tag{3.15}
$$

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The path is assumed to enter into Heaven immediately after t_0, *i.e.*, at $t = t_1 = t_0 + \epsilon$ with any infinitesimal ϵ, we have

$$Q_2(t_1) > 0, \quad Q_3(t_1) > 0, \quad \cdots, \quad Q_n(t_1) > 0. \tag{3.16}$$

In the following we argue that the answer to the question is negative, that is, there exists no such path for some n.

For this purpose it is useful to consider the two steps, described below. Calling the point

$$Q_2(t_0) = Q_3(t_0) = \cdots = Q_n(t_0) = 0 \ , \tag{3.17}$$

'gate', we wish to argue along the line of the following two statements:

(i) *There exists no path entering from Hell to Heaven through the gate.*

(ii) *Any path which enters into Heaven from Hell must pass the gate.*

Let us prove (i) for $n = 2$ mod 4, and $n = 3$. For this, we observe first that if (3.17) holds then $0 = Q_n(t_0) = u_{2n}(t_0)$ and $0 = Q_{n-1}(t_0) = -u_{2n-1}^2(t_0)$, that is, we get $u_{2n}(t_0) = u_{2n-1}(t_0) = 0$. Repeating this process we find that (3.17) actually means

$$u_{n+2}(t_0) = u_{n+3}(t_0) = \cdots = u_{2n}(t_0) = 0, \tag{3.18}$$

i.e., all the entries of g lower than the anti-diagonal line are zero. Hence, evaluating the determinant of g at $t = t_0$ we get

$$1 = \det g(t_0) = (-1)^{P[n]} u_{n+1}^n(t_0), \tag{3.19}$$

where $P[n] := n(n-1)/2$ is the factor of permutation attached. This result (3.19) shows that, for $n = 2$ mod 4, there is no real solution for $u_{n+1}(t_0)$, that is, the gate point (3.17) does not even exist in the reduced phase space M_{red}. This concludes the proof for $n = 2$ mod 4. Note that the gate point does exist in M_{red} in other cases with the solutions

$$u_{n+1}(t_0) = \begin{cases} \pm 1, & \text{if } n = 0 \text{ mod } 4; \\ +1, & \text{if } n = 1 \text{ mod } 4; \\ -1, & \text{if } n = 3 \text{ mod } 4. \end{cases} \tag{3.20}$$

Thus, to argue for other cases we need something more.

Now we consider the case $n = 3$. From (3.16) we must have

$$Q_3(t_1) = u_6(t_1) > 0 \ , \qquad Q_2(t_1) = u_4(t_1)u_6(t_1) - u_5^2(t_1) > 0 \,. \tag{3.21}$$

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On the other hand, on account of the smoothness assumption we made for the path, the condition $u_4(t_0) = -1$ in (3.20) implies $u_4(t_1) < 0$, which contradicts with (3.21) above. We therefore have shown the statement (i) for $n = 3$. One can prove also for $n = 4$ using a slightly involved but similar argument, for which we refer to [8]. The statement (i) has not been (dis)proven for a generic n, yet.

Since the second statement (ii) seems even harder to argue in general, we prove here for $n = 3$ only (the case $n = 2$ is trivial), and again for $n = 4$ we refer to [8]. Suppose that there exists a path entering from Hell to Heaven without passing the gate. The possibilities are thus either

$$Q_3(t_0) = 0 \quad \text{and} \quad Q_2(t_0) > 0 \tag{3.22}$$

or

$$Q_3(t_0) > 0 \quad \text{and} \quad Q_2(t_0) = 0. \tag{3.23}$$

The first possibility (3.22) can be denied at once since it is inconsistent with the definitions for Q_2 and Q_3 (cf.(3.21)). To deny (3.23), we just use $0 = Q_2(t_0) = u_4(t_0)u_6(t_0) - u_4^2(t_0)$ to obtain

$$0 < Q_3^3(t_0) \det g(t_0) = u_6^3(t_0) \det \begin{pmatrix} g_{11} & g_{12} & u_4 \\ g_{12} & u_4 & u_5 \\ u_4 & u_5 & u_6 \end{pmatrix} (t_0)$$
$$= -\left\{ u_5^3(t_0) - g_{12}(t_0)u_6^2(t_0) \right\}^2 , \tag{3.24}$$

which is, again, a contradiction. Thus, combining the result obtained earlier, we have learned that for $n = 2$, 3 and 4, the Toda subsystem is disconnected from all the rest of the subsystems in the reduced system.

4. $SL(2, \mathbf{R})$ Toda (Liouville) theory in $1 + 1$ dimensions

Although it is not simple to discuss the generic $SL(n, \mathbf{R})$ case, we can proceed similarly for the $1 + 1$ dimensional field theory case at least for $SL(2, \mathbf{R})$, $i.e.$, for the WZNW $\rightarrow$ Liouville reduction. Our study of the global aspects in this field theory case then reveals an intricate structure that we did not find in the $0 + 1$ dimensional counterpart in sect.2. The relation to the singularity in classical solutions will also be discussed at the end.

4.1. Hamiltonian reduction

We first recall the WZNW $\rightarrow$ Liouville reduction starting, as in the $0 + 1$ dimensional case in sec.2, with the Hamiltonian description of the WZNW model. Consider the phase space of the type (2.1) but with $(g(x^1), J(x^1))$ both periodic functions in space x^1, and postulate the fundamental Poisson brackets,

$$\{g_{ij}(x^1), g_{kl}(y^1)\} = 0,$$
$$\{g_{ij}(x^1), \mathrm{tr}(T^a J(y^1))\} = (T^a g(x^1))_{ij}\, \delta(x^1 - y^1),$$
$$\{\mathrm{tr}(T^a J(x^1)), \mathrm{tr}(T^b J(y^1))\} = \mathrm{tr}([T^a, T^b] J(x^1))\, \delta(x^1 - y^1) + 2\,\mathrm{tr}(T^a T^b)\, \delta'(x^1 - y^1),$$
$$(4.1)$$

where $\delta' = \partial_1 \delta(x^1 - y^1)$. The Hamiltonian is then taken to be

$$H = \int dx^1\, \frac{1}{4}\mathrm{tr}(J^2 + \tilde{J}^2), \tag{4.2}$$

which yields the field equations

$$\dot{g} = \{g, H\} = Jg - g', \qquad \dot{J} = \{J, H\} = J', \tag{4.3}$$

or $\partial_-(\partial_+ g\, g^{-1}) = 0$. The right-current, which acts as the generator for the right-transformation, reads

$$\tilde{J} = -g^{-1} Jg + 2g'g^{-1}. \tag{4.4}$$

Upon imposing the constraints (2.7), which are still first class under the Poisson brackets (4.1), we have the gauge transformations generated by them,

$$g \longrightarrow \alpha\, g\, \gamma, \qquad J \longrightarrow \alpha\, J\, \alpha^{-1} + 2\alpha'\alpha^{-1}. \tag{4.5}$$

In chiral currents the constrains (2.7) are formally the same as before, but in components they are not quite so — the first constraint in (2.7) is unchanged $j_- = \mu$ whereas in view of (4.4) the second one now takes the form,

$$\mu g_{12}^2 - 2j_0\, g_{22}g_{12} - j_+ g_{22}^2 + 2(g_{22}g_{12}' - g_{12}g_{22}') = -\nu. \tag{4.6}$$

At first sight it appears that the additional term in (4.6) does not alter the condition for disconnectedness of the reduced phase space observed in the toy model. However, due to the space dimension x^1 the previous argument must be modified. In fact, we cannot rule out now the possibility of having $g_{22}(x^1) = 0$ with $g_{22}'(x^1) \neq 0$ at *some*

points of x^1 (not all x^1), in which case the above equation has a solution for $g_{12}(x^1)$. In other words, the value of g_{22} could go out of the patches $\tau = \pm 1$ in some domain of x^1. Thus we see that the description of the Gauss decomposable patch is not sufficient *irrespective* of the sign of $\mu\nu$. An interesting observation is available at this point: since for the case $\mu\nu > 0$ the configurations $g_{22} = 0$ *and* $g'_{22} = 0$ are incompatible at any point x^1, one cannot shrink or extend the loop of g_{22} — the periodic condition in x^1 implies that the configuration of g_{22} can be expressed by a loop in G — *across* the border of the two patches $\tau = \pm 1$. This shows that g_{22} has a conserved topological charge given by the number of zeros $g_{22}(x^1) = 0$ over the period in x^1. The meaning of the charge will be discussed later. Hereafter, we shall consider only for the case $\mu\nu > 0$ (which is also the case of interest from two dimensional gravity point of view) and, for simplicity, we shall set $\mu = \nu = 1$.

As in the toy model, we now try to give a global Hamiltonian description in terms of local Hamiltonian subsystems. Again, it is easy to find a set of gauge invariant differential polynomials; one only has to change H in (2.19) slightly,

$$Q = g_{22}, \qquad P = g_{12} - j_0 g_{22}, \qquad V = j_0^2 + j_+ - 2j_0'. \tag{4.7}$$

They form under the Poisson brackets the following differential polynomial algebra:

$$
\begin{aligned}
\{Q\,,Q\} &= 0, & \{Q\,,P\} &= \frac{1}{2}Q^2\delta, \\
\{P\,,P\} &= \frac{1}{2}(Q^2)'\delta + Q^2\delta', & \{Q\,,V\} &= P\delta - Q\delta', \\
\{P\,,V\} &= VQ\delta + P\delta' - Q\delta'', & \{V\,,V\} &= 2V'\delta + 4V\delta' - 4\delta''',
\end{aligned}
\tag{4.8}
$$

where $\{Q\,,Q\} = \{Q(x^1)\,,Q(y^1)\}$ and so on. Note that besides the Virasoro subalgebra formed by V there exists another differential polynomial subalgebra formed by (Q, P). Also, since we have

$$\dot{Q} = \{Q\,,H\} = P - Q', \tag{4.9}$$

or $P = \partial_+ Q$, the fourth equation in (4.8) shows that Q is a conformal primary field of weight $-\frac{1}{2}$. As in the toy model the three gauge invariant functions are not independent but satisfy

$$VQ^2 = P^2 - 2Q'P + 2QP' + 1. \tag{4.10}$$

This relation (which is in fact idential to (4.6)) may define the reduced phase space as a hypersurface in the space spanned by $(Q(x^1), P(x^1), V(x^1))$.

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In particular, for $Q \neq 0$ it is convenient to write

$$Q = \pm e^{-\frac{\phi}{2}}, \tag{4.11}$$

and define the momentum conjugate to ϕ as $\pi = \frac{1}{Q}(Q' - P)$. With these variables the Virasoro density V takes the familiar form of the Liouville theory,

$$\begin{aligned} V &= (\pi + \frac{1}{2}\phi')^2 - 2(\pi + \frac{1}{2}\phi')' + e^{\phi} \\ &= \frac{1}{4}(\partial_+\phi)^2 - (\partial_+\phi)' + e^{\phi}. \end{aligned} \tag{4.12}$$

One can derive the (global) field equation for Q by proceeding further from (4.9),

$$\partial_+^2 Q = VQ \qquad \text{or} \qquad \partial_+ Q \partial_- Q - Q\, \partial_+ \partial_- Q + 1 = 0, \tag{4.13}$$

where we have used (4.9), (4.10) to obtain the second form of the field equation. Of course, for $Q \neq 0$ (4.13) reduces to the Liouville equation,

$$\partial_+ \partial_- \phi + 2e^{\phi} = 0, \tag{4.14}$$

in the variable of (4.11). In summary, we find that the reduced WZNW theory contains the Liouville theory locally, and the reduced WZNW theory consists of two copies of the Liouville theory in the patches $\tau = \pm 1$ (*i.e.*, $Q > 0$ and $Q < 0$) glued together by the domain-wall $Q = 0$.

4.2. Global classical solution and the index of singularity

From the above construction and the relation between Q and ϕ we expect, by using the globally defined Q, that the singularities in the solution ϕ of the Liouville theory disappear and moreover they may be classified by means of the topological charge carried by Q. Let us examine these points through the solution of the global equation (4.13) next.

The key observation for getting the solution of the partial differential equation (4.13) is the one we employed to solve the Liouville equation in the WZNW context: it can be obtained from the WZNW solution, which is trivial, by taking into account the constraints. In our case, since Q is g_{22} itself the WZNW solution $g(x) = g^L(x^+)g^R(x^-)$ implies

$$Q = g_{22} = g_{21}^L\, g_{12}^R + g_{22}^L\, g_{22}^R \,. \tag{4.15}$$

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From the fact that $J = \partial_+ g^L \, (g^L)^{-1}$ and $\tilde{J} = -(g^R)^{-1} \partial_- g^R$ at on-shell it follows that the constraints (2.7) are equivalent to

$$\partial_+ g_{21}^L \, g_{22}^L - \partial_+ g_{22}^L \, g_{21}^L = 1,$$
$$\partial_- g_{12}^R \, g_{22}^R - \partial_- g_{22}^R \, g_{12}^R = 1. \tag{4.16}$$

This means that instead of the original second order partial differential equation (4.13) we only have to deal with the two linear ordinary differential equations (4.16), which are easy to solve. Let us give a quick way to reach the solution here. The structure shared by both of the equations is $f'g - fg' = 1$. For $fg \neq 0$ we rewrite it as $(\ln \frac{f}{g})' = \frac{1}{fg}$, which is integrated to be

$$\frac{f}{g} = e^{\int \frac{dx}{fg} + c} \equiv F(x), \tag{4.17}$$

where c is a constant of integration and $F(x)$ is defined by this equation, which we regard as an arbitrary function. A differentiation of F gives $F' = \frac{1}{g^2}$, which shows that we must have $F' > 0$. The solution is then given by

$$f = \pm \frac{F}{\sqrt{F'}}, \qquad g = \pm \frac{1}{\sqrt{F'}}. \tag{4.18}$$

Applying this procedure to the equations (4.16), and combining with (4.15), we get the solution for Q:

$$Q(x) = \pm \frac{1 + F(x^+)G(x^-)}{\sqrt{\partial_+ F(x^+) \partial_- G(x^-)}}, \tag{4.19}$$

where $F(x^+)$ and $G(x^-)$ are arbitrary functions of the argument with $\partial_+ F > 0$, $\partial_- G > 0$, such that Q be periodic in x^1. For instance, if we set the periodic condition $Q(x^0, x^1 + 2\pi) = Q(x^0, x^1)$, the choice, $F(x^+) = \tan u x^+$, $G(x^-) = \tan v x^-$ with positive constants u, v, satisfies the periodicity condition if $\frac{1}{2}(u+v) = n$ is an (positive) integer. Putting $\frac{1}{2}(u - v) = r$ with $-n < r < n$ we find the solution

$$Q(x) = \pm \cos(rx^0 + nx^1). \tag{4.20}$$

On the other hand, the global solution (4.19) reduces on the patches $\tau = \pm 1$ to the local one, namely the well-known Liouville solution,

$$\phi(x) = \ln \left(\frac{\partial_+ F(x^+) \partial_- G(x^-)}{[1 + F(x^+)G(x^-)]^2} \right). \tag{4.21}$$

The previous special solution (4.20) therefore leads to

$$\phi(x) = -2 \ln |\cos(rx^0 + nx^1)|. \tag{4.22}$$

We then notice that there appears in the special solution essentially the same property observed in the toy model — the apparently singular motion in the local system is merely a part of the oscillation in the global system. In general, the global dynamics of physical quantities in the present model could turn out to be regular despite that the local counterpart might appear singular.

Finally, let us detail the point of singularity slightly more. From the above analysis we learn that to any global (non-singular) solution Q there exists a local (possibly singular) Liouville solution ϕ and that the converse is also true. This implies that the singularity in the Liouville solution is characterized by the zeros of Q. More precisely, if we define the singularity index s of a Liouville solution by the number of singular points at $x^0 = $ constant surface, then s is equal to the number of zeros of Q at the fixed time. But since the number of zeros of Q is the topological charge which is *time-independent*, it can be used to classify the Liouville solution. For example, the previous special solution (4.20) has index $s = 2n$ (in our case s is always an even integer due to the periodic condition). On the other hand, since (4.15) is a linear differential equation, one can in fact construct a Liouville solution with any number of singular points s by imposing an appropriate initial condition for Q with s zeros.

5. Conclusions and outlooks

We have seen in this paper that the reduced theory obtained by the WZNW $\rightarrow$ Toda reduction is not just the standard Toda theory but has richer global structures which may bring drastic changes in the interpretation of the dynamics. More precisely, it was shown that the reduced system actually consists of many quasi-Toda subsystems where the (dis)connectedness among them is crucial in determining the global effects on the dynamics. We also found that the $1 + 1$ dimensional 'global Liouville theory' constructed by the WZNW reduction possesses a conserved topological charge which classifies the classical solutions. Our general framework for studying the global aspects of the WZNW $\rightarrow$ Toda reduction uses the Bruhat decomposition and the vector DS gauge, both of which allow an immediate generalization to any other simple Lie groups, not just to $SL(n, \mathbf{R})$.

These results, although still preliminary, suggest that those theories obtained by the WZNW reduction — whether or not they are in $0 + 1$ dimension or $1 + 1$ dimensions — are interesting enough physically and worth further investigation. Some of the possible

directions are as follows:

(1) The analysis of the (dis)connectedness of the subsystems in the reduced theory is far from complete, since we know only for $SL(n, \mathbf{R})$ with $n = 2$, 3, 4 that the proper Toda subsystem is disconnected from the rest. It seems however reasonable to conjecture that it is disconnected for any n. Of course, we would like to have full information on the global structure of the reduced theory, *e.g.*, as to how all those subsystems are glued together for any possible signs of μ_α, ν_α.

(2) More importantly, the change of the interpretation in the nature of dynamics by being global could imply a drastic change in the theory at the quantum level as well. An interesting question is whether it is possible to construct a reasonable quantum gravity model in two dimensions by quantizing the $1 + 1$ dimensional global Liouville theory obtained by the WZNW reduction in the non-trivial topological sectors. The difficulty is that in these sectors the energy is not bounded from below in general [9], similar to the case of the quasi-Toda subsystems (3.6) of the toy model for generic m. In the $SL(2, \mathbf{R})$ toy model, a consistent quantum mechanical version of the reduced system has been recently constructed [10] in the topologically non-trivial case $\mu\nu < 0$.

(3) Our classification of the Liouville solutions by their topological charge is related to earlier work [11] on the singular sectors of the Liouville theory. The dependence of the topological charge on the 'coadjoint orbit type' of the corresponding chiral Virasoro densities is analysed in detail in [9], generalizing results of [12]. The precise relationships between these results as well as the possible connections between the globally well defined reduced WZNW systems and the $\mathcal{W}$-geometry theories proposed in [13,14] as geometric reformulations of Toda theories deserve further study.

We hope that the present paper has provoked the interest of the reader. More technical accounts of the global aspects of the reduced WZNW systems will appear elsewhere.

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